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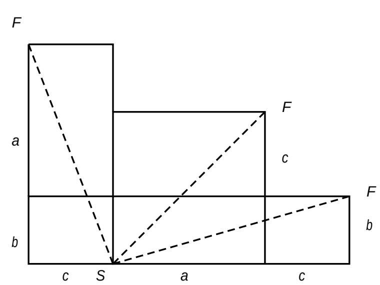
MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 6, June 5, 2016

1. If we atten out the cuboid, we can see more clearly the paths along the walls the spider can take to get to the y. Since the shortest distance between two points is always a straight line (consequence of the triangles inequality), we see that there are three potential ways the spider can get to the y; as shown below. The distance the spider travels in each way is



$$d_1 = \sqrt{c^2 + (a+b)^2}$$
$$d_2 = \sqrt{a^2 + (b+c)^2}$$
$$d_3 = \sqrt{b^2 + (a+c)^2}.$$

With out loss of generality, we can assume that $a > c > b$. Then d_2 is the shortest, since $d_2 = \sqrt{a^2 + (b+c)^2} = \sqrt{a^2 + b^2 + c^2 + 2bc}$, and $bc < ac < ab$.

2. First we work out how many strictly increasing numbers there are. If we list the digits $1, 2, 3, \ldots, 9$ in order, then we can make a strictly increasing number by selecting one or more digits from the list. There are 2

where we have used the fact that $ac - a - 1 = c$ on the second equality. From (2), one has $c = 3$, $a = 2$ or $a = 3; c = 2$. Moreover, from (1), if $a = 2; 3$ then $b = 6$. Hence, $2 \le x < 4$ and $y = b - x = 6 - x$. But x and y are both not integers in this case, so we eliminate $x = 4$ from our solution.

Thus, we have three possible solutions: $2 < x < 3$, $y = 6 - x$ or $3 < x < 4$, $y = 6 - x$ or $x = y = 2$.

Senior Questions

1. Let w_{ij} be the result of the game between the i-th and j-th players:

$$s_{ij} = \begin{cases} +1 & \text{if player } i\text{-th wins,} \\ -1 & \text{if player } j\text{-th wins,} \\ 0 & \text{if they have a draw or } i = j. \end{cases}$$

Let X_j be the total score of player j at the end of the tournament. Then the number

$$\sum_{j=1}^{n} s_{ij} X_j,$$

is the difference between the total score of those players who beats player i and those who were beaten by player i, for some fixed $1 \le i \le n$. Thus, the question is asking whether $\sum_{j=1}^{n} s_{ij} X_j > 0$ for every i. Consider

$$\sum_{i,j=1}^{n} s_{ij} X_j X_i. \qquad (4)$$

The summation (4) is equal to 0, since $s_{ij} = -s_{ji}$. But $X_i \ge 0$ for each i, therefore (4) implies $\sum_{j=1}^{n} s_{ij} X_j \not> 0$. So, it is impossible.

2. Let us arrange all the students in the school according to the number of "A" marks they received. So, $A_1 \ge A_2 \ge \dots \ge A_n$ where A_j is the number of "A" received by j-th student, $1 \le j \le n$, $A_j \ge 0$ and $\sum_{j=1}^{n} A_j = A$ where A is a total number of "A marks.

Now let us consider the first five students. According to the condition, one student (who has to be on top of the list) got at least 80% of "A" marks received by this group, which leaves no more than 20% of "A" marks remaining for the other four students. So, $A_2 + A_3 + A_4 + A_5 \le \frac{1}{4}A_1$, and we have an estimate $A_2 \le \frac{1}{4}A_1$. Considering students from k-th to $k+4$-th ($k + 4 \le n$), we conclude that $A_{k+1} \le \frac{1}{4}A_k$, which implies that $A_{k+1} \le \frac{1}{4^k}A_1$ ($k \le n - 5$) and $A_{n-3} + A_{n-2} + A_{n-1} + A_n \le \frac{1}{4}A_{n-4}$. Now we have

$$\begin{aligned} A &= A_1 + A_2 + \dots + A_{n-4} + (A_{n-3} + \dots + A_n) \\ &\le A_1 + \frac{1}{4}A_1 + \frac{1}{4^2}A_1 + \dots + \frac{1}{4^{n-5}}A_1 + \frac{1}{4^{n-4}}A_1 \\ &< \sum_{k=0}^{\infty} \frac{1}{4^k}A_1 = \frac{A_1}{1 - \frac{1}{4}} = \frac{4}{3}A_1 \end{aligned}$$

Therefore, $A_1 > \frac{3}{4}A$.

3. One ways is to used induction, the identity $\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{3+\sqrt{5}}{2}$ may be helpful. The other way is the treat it as a first order difference equation.