MATHEMATICS ENRICHMENT CLUB.

Solutions Sheet 8, June 16, 2015[1]

1. We have

$$\begin{aligned} x^2 - y^2 &= 1999 \\ (x - y)(x + y) &= 1999; \end{aligned}$$

   and since 1999 is prime, either $(x - y) = \pm 1$, $(x + y) = \pm 1999$ or $(x + y) = \pm 1$, $(x - y) = \pm 1999$; there is a total of four integral solutions to both cases.

2. There are two solutions to this problem: One uses the perimeter of $\triangle ABC$, the other uses the area of $\triangle ABC$.

   (a) Hint: Let the point of intersection between the circle and the side AB be P. Then the radius of the inscribed circle is $|AP|$, and the line PB is tangent to the circle at the point P.

   (b) Hint: Let M be the middle of the inscribed circle. Then the triangles $\triangle ABM$, $\triangle BCM$ and $\triangle CAM$ all have height equal to the radius of the inscribed circle.

3. A neat trick is to express N as

$$\underbrace{333\ldots333}_{61\ 3's} = \frac{3}{9}\left(\underbrace{999\ldots999}_{61\ 9's}\right) = \frac{3}{9}(10^{61} - 1).$$

   Similarly, $M = \underbrace{666\ldots666}_{62\ 6's} = \frac{6}{9}(10^{62} - 1)$. Now

$$\begin{aligned} N \cdot M &= \frac{2}{9}(10^{61} - 1)(10^{62} - 1) \\ &= \frac{2}{9}(10^{61} - 1) \cdot 10^{62} - \frac{2}{9}(10^{61} - 1) \\ &= \underbrace{222\ldots222}_{61\ 2's}\underbrace{000\ldots000}_{62\ 0's} - \underbrace{222\ldots222}_{61\ 2's} \\ &= \underbrace{222\ldots222}_{60\ 2's}19\underbrace{777\ldots777}_{60\ 7's}8. \end{aligned}$$

[1] Some problems from UNSW's publication *Parabola*.

It is easy to to compute the sum of digits on the last line of the above equation; it is 558.

4. First we note that for positive integers $m; n; k$ and r, if $a = mk + r$, then $a^x = nk + r^x$ (you may want to show this is true). Now, since $a = x$

then by using the assumption that $x < y$, we have

$$d\ \frac{x\ \ y}{2}\quad 1\ \ \frac{y}{x}^{\ 1} = \frac{x}{2}$$
$$d\ \frac{x\ \ y}{2}\quad 1\ \ \frac{x}{y}^{\ 1} = \frac{y}{2}:$$

The last system of inequality does not hold because $x < y$, so we have a contradiction to the advertisement's claim.

Senior Questions

1. Since $\ > 0$, $\ + \frac{1}{\ }^{2} = \ ^{2} + \frac{1}{\ ^{2}} + 2\quad 2$. Similarly, $\ + \frac{1}{\ }^{2}\quad 2$. Therefore, if r_1 and r_2 are the roots of f (assuming $r_1\quad r_2$ wlog). Then $r_1\quad 2$ and $r_2 < 0$, so that $r_1 r_2 = c\quad 3 < 0$, which implies $c < 3$.

   To get the lower bound on c, we use the quadratic formula $2\quad r_1 = (c+1) + \sqrt{(c+1)^2\quad 4(c\quad 3)}$. Solving gives $\ 2\quad c$.

2. Lets start by looking at the extreme case $BX = XC; CY = YA; AZ = ZB$; as shown below. By the Midpoint Theorem, the line BC is parallel to ZY, and the line AC is parallel to ZX

Since the RHS of the above equation is rational, $\sqrt{c}$ must be rational. Write $\sqrt{c} = x/y$, where x and y are integers with greatest common multiplier one. Then $c = x^2/y^2$, and greatest common multiplier between x^2 and y^2 is one. Since c is an integer, x^2 must