

MATHEMATICS ENRICHMENT CLUB. 1
 Solution Sheet 6, June 11, 2013

1. The prime factorisation of $770 = 2 \cdot 5 \cdot 7 \cdot 11$, so assuming by adults we mean over 18 year olds, our two people are 22 and 35.
2. (Disclaimer: Introduction 'group theory' answer - this question can be answered more simply by deductive logic, or guess and check (maximum 13 guesses), but this question's close ties to group theory I think warrants a bit of abstract algebra. If you just want the answer, skip to the end©)

Let's write the card shu er as a function σ , where $\sigma(n)$ is the new position of the n th card after one shuffle. We'll also write iterated shuffles as σ^m , meaning m compositions of the shuffling function. As a final piece of notation, we'll introduce 'k-cycles', which are written as a collection of numbers in a pair of brackets and indicate that the value of each number is that to its immediate right (or the first position if at the end of the cycle), e.g. $(1\ 2\ 3)$ means $1! \rightarrow 2, 2! \rightarrow 3$ and $3! \rightarrow 1$.

The information given tells us

$$\sigma^2 = (1\ 12\ 5\ 2\ 7\ 9\ 11\ 10\ 4\ 13\ 3\ 8\ 6)$$

We can multiply (compose) cycles together just by tracing from left (i.e. applying the cycles to each number left to right), for example

$$(1\ 2\ 3)(2\ 1\ 4) = (1)(2\ 3\ 4) = (2\ 3\ 4)$$

since $1! \rightarrow 2, 2! \rightarrow 1, 2! \rightarrow 3, 3! \rightarrow 1, 4! \rightarrow 4$ and $4! \rightarrow 2$.² In this manner we can repeatedly multiply σ^2 and we find

$$\sigma^{26} = (1)(2)(3)(4)(5)(6)(7)(8)(9)(10)(11)(12)(13);$$

i.e. shuffling 26 times puts the cards back in to the order they originally were. This means the 'order' of σ is 26, where the 'order' of a permutation is how many times

¹Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres, Macquarie Uni.

²An interesting result is that every permutation can be written as a product of 2-cycles, e.g. $(1\ 2\ 3) = (1\ 3)(3\ 2)$, and even though this 2-cycle representation is not unique, it is always made up of either an odd or even number of 2-cycles.

you multiply it by itself to get the identity function - one that leaves everything alone like the one above.

Since σ is, at most, a 13-cycle its order is ≤ 13 . So the order of σ^{12} could be 1, 2, 13 or 26 in order to satisfy $\sigma^{26} = \text{id}$, but it can't be 26, it's not 1 or 2 from the given information, so it must have order 13.

So now we work out σ^{12} , then we can determine σ so that $\sigma^{12} = \text{id}$. I worked out σ^{12} by first performing

$$\sigma^2 = (1\ 5\ 7\ 11\ 4\ 3\ 6\ 12\ 2\ 9\ 10\ 13\ 8)$$

then

$$\sigma^8 = \sigma^4 \sigma^4 = (1\ 7\ 4\ 6\ 2\ 10\ 8\ 5\ 11\ 3\ 12\ 9\ 13)$$

and finally

$$\sigma^{12} = \sigma^8 \sigma^4 = (1\ 11\ 6\ 9\ 8\ 7\ 3\ 2\ 13\ 5\ 4\ 12\ 10)$$

To find σ I then wrote it as a 2-cycle representation

$$\sigma^{12} = (a\ 1)(b\ 2)(c\ 3)(d\ 4) \dots (m\ 13)$$

and work through, from left to right, making sure I put the numbers back where they started. For instance $\sigma^{12}(1) = 11$, so set $a = 11$, $\sigma^{12}(2) = 13$, so $b = 13$, $\sigma^{12}(3) = 2$ so $c = 2$ (I've already made $b = 13$, and so far so good! $\sigma^{12}(4) = 13$ so now I make $13 \rightarrow 3$ after, so that overall $2 \rightarrow 13$). Continuing, we find

$$\begin{aligned} \sigma &= (11\ 1)(13\ 2)(13\ 3)(12\ 4)(12\ 5)(9\ 6)(13\ 7)(13\ 8)(13\ 9)(11\ 10)(11\ 12)(11\ 13) \\ &= (1\ 10\ 12\ 4\ 5\ 13\ 2\ 3\ 7\ 8\ 9\ 6\ 11) \end{aligned}$$

Finally, this means the cards originally ordered A; 2; 3; 4; 5; 6; 7; 8; 9; 10; J; Q; K become, after one shuffle, J; K; 2; Q; 4; 9; 3; 7; 8; A; 6; 10; 5.

3. (a) Draw the right angled triangle ABC with right angle at C. Let D be the midpoint of AB, and E a point on AC such that AC = 2 DE. Then $\triangle ADE$ is similar to $\triangle ABC$ (three angles equal). Since $AD = \frac{1}{2}AB$ then $AE = \frac{1}{2}AC$ or rather $AE = EC$. Now $\triangle AED$ is congruent to $\triangle CED$ (two sides equal, $AE = EC$, DE common, and an included angle $\angle AED = \angle DEC$). Thus $\frac{1}{2}AB = AD = DC$.
 - (b) From part i) we see $DB_1 = B_1C$ and $DC_1 = C_1B$. Note that $\triangle CB_1A_1$ is similar to $\triangle CAB$ (two sides in ratio and an included angle). The sides are in ratio 1 : 2 so $A_1B_1 = \frac{1}{2}AB = C_1B$, and so $A_1B_1 = DC_1$. Similarly $\triangle BC_1A_1$ is similar to $\triangle BAC$, so $C_1A_1 = B_1C = B_1A_1$. Thus $\triangle B_1C_1D$ and $\triangle B_1C_1A_1$ are congruent because they have 3 equal sides.
4. Following the hint, we must have $3m - 1 = n$ or $3m - 1 = 2n$, since $3m - 1 < 3n$. So

$$3(3m - 1) - 1 = km; \quad k \in \mathbb{Z}$$

$$(9 - k)m = 4$$

$$m = \frac{4}{9 - k}$$

$$m = 4; 2; \text{ or } 1;$$

or

$$\begin{aligned} 3 \frac{3m-1}{2} &= km \\ 9m-3 &= 2km \\ m &= \frac{5}{9-2k} \\ m &= 5; \text{ or } 1: \end{aligned}$$

Thus the pairs are (1; 1), (1; 2), (2; 5), (4; 11) and (5; 7).

5. (a) $(12) = 4, (30) = 8$

(b) We can think of $\phi(n)$ as being the number of numbers less than n which are not a multiple of a factor of n (except the factor 1). So if p is prime, its only factors are 1 and p , so every other number is not a multiple of a factor that isn't 1, except itself. Thus $\phi(p) = p - 1$.

For p^2 , the factors are 1, p and p^2 , so the multiples of the factors that aren't 1 are $p; 2p; 3p; \dots; p^2$, of which there are p . So $\phi(p^2) = p^2 - p$.

For p^3 , the factors are 1, $p; p^2$ and p^3 , so the multiples of the factors that aren't 1 are $p; 2p; 3p; \dots; p^2; (p+1)p; \dots; 2p^2; (2p+1)p; \dots$, that is, the multiples of p^2 are contained in the multiples of p , of which there are p^2 . So $\phi(p^3) = p^3 - p^2$.

(c) Using the same method as above, the factors of pq are 1, $p; q$ and pq , so the multiples of the factors that aren't 1 are $p; 2p; 3p; \dots; qp$ (q of them) and $q; 2q; 3q; \dots; pq$ (p of them), but we don't want to count pq twice. So $\phi(pq) = pq - q - p + 1$.

6. We use the fact that the medians divide $\triangle ABC$ into 2 equal area pieces, and that S is $\frac{2}{3}$ along the median from A (you can prove these by considering the areas of smaller triangles with the same heights).

Let the median from A meet BC at P , since ST is parallel to BC triangles APC and AST are similar - 3 angles equal. Since $AS = \frac{2}{3}AP$ then the area of AST is $\frac{4}{9}$ the area of APC which is half the area of ABC so the area of AST is $\frac{2}{9}$ the area of ABC .

Senior Questions

1. Let $f(x) = 2x^n - nx^2 + 1$, then $f'(x) = 2nx(x^{n-2} - 1)$. So f has stationary points at $x = 0$ and $x = 1$ (since $n > 3$ and odd). Taking the second derivative $f''(x) = 2n(n-1)x^{n-2} - 2n$, so $f''(0) = -2n < 0$ and $f''(1) = 2n(n-1) - 2n = 2n(n-2) > 0$. So $x = 0$ is a local max and $x = 1$ is a local min.

Finally $f(0) = 1 > 0$ and $f(1) = 3 - n < 0$. Since these are the only stationary points, f is monotonic between/outside of them. Since $x = 0$ is a local max, and positive there is one root for $x < 0$, which is unique since f is monotonic decreasing for $x < 0$. Since $f(0) > 0 > f(1)$ and f is monotonic between 0 and 1 there is exactly one root for $0 < x < 1$. Since $x = 1$ is a local min, $f(1) < 0$ and $f(x)$ is monotonic increasing for $x > 1$ there is exactly one root for $x > 1$. Thus, in total, there are 3 roots.

2. Take the log of both sides and differentiate both sides with respect to x .

$$\log f(x) = x \log \left(1 + \frac{1}{x} \right)$$

$$\frac{f'(x)}{f(x)} = \log \left(1 + \frac{1}{x} \right) - \frac{1}{x+1}:$$

3. Draw the graph of $y = \frac{1}{t}$ for t between 1 and $1 + \frac{1}{x}$ and we see that the area under the curve is larger than the area of the rectangle with base $1 + \frac{1}{x} - 1$ and height $\frac{1}{1 + \frac{1}{x}}$, so

$$\int_1^{1 + \frac{1}{x}} \frac{1}{t} dt = \log \left(1 + \frac{1}{x} \right) > \frac{1}{x} \frac{1}{1 + \frac{1}{x}} = \frac{1}{1 + x}:$$

Thus $\frac{f'(x)}{f(x)} > 0$, and since $f(x) > 0$ for all x so is $f'(x)$.